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A note on twist two operators in $\mathcal{N}=4$ SYM and Wilson loops in Minkowski signature

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Abstract: Recently, the anomalous dimension of twist two operators in $\mathcal{N}=4$ SYM theory was computed by Gubser, Klebanov and Polyakov in the limit of large 't Hooft coupling using semi-classical rotating strings in AdS_5. Here we reproduce their results for large angular momentum by using the cusp anomaly of Wilson loops in Minkowski signature also computed within the AdS/CFT correspondence. In our case the anomalous dimension is related to an Euclidean world-sheet whose properties are completely determined by the symmetries of the problem. This gives support to the proposed identification of rotating strings and twist two operators.

Keywords: Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

The AdS/CFT correspondence [1]–[3] provides a description of the large N limit of $\mathcal{N} = 4$ SU(N) SYM theory as type IIB string theory on $AdS_5 \times S_5$. This has led to much progress in the understanding of SYM theories in the 't Hooft limit [4]. However the precise correspondence between states in $AdS_5 \times S_5$ and operators in the SYM theory is not known in general. Most developments[1] are based on the relation between chiral primary operators in the SYM theory and supergravity modes in $AdS_5 \times S_5$, as well as in the relation between Wilson loop operators and semiclassical world-sheets in $AdS_5 \times S_5$. In [5], Berenstein, Maldacena and Nastase extended the correspondence to a set of operators in the SYM theory corresponding to excited string states in $AdS_5 \times S_5$ with large angular momentum along the S_5. In a related development, Gubser, Klebanov and Polyakov [6][2] suggested that states with angular momentum in the AdS_5 part should correspond to twist two operators in the SYM theory. In free field theory these are the operators with lowest conformal dimension for a given spin S one example being:

$$\mathcal{O}_{\{\mu_1\cdots\mu_S\}} = \mathrm{Tr}\,\Phi^I \nabla_{\{\mu_1} \cdots \nabla_{\mu_S\}} \Phi^I \,, \tag{1.1}$$

where Φ^I are the scalars of the $\mathcal{N} = 4$ theory, and the indices $\{\mu_1 \cdots \mu_S\}$ are symmetrized and traces removed so that the operator carries spin S. The conformal dimension in free field theory is $\Delta_S = S + 2$, and the twist $\Delta_S - S = 2$. Other twist two operators can be constructed using the gauge field or the fermions instead of the scalar fields. The interest of these operators is that in QCD they give the leading contribution to deep inelastic scattering [9]. The proposal of [6] is that such operators can be described, in the dual

[1] See [4] for a review and a complete set of references.

[2] See also the recent works [7, 8].

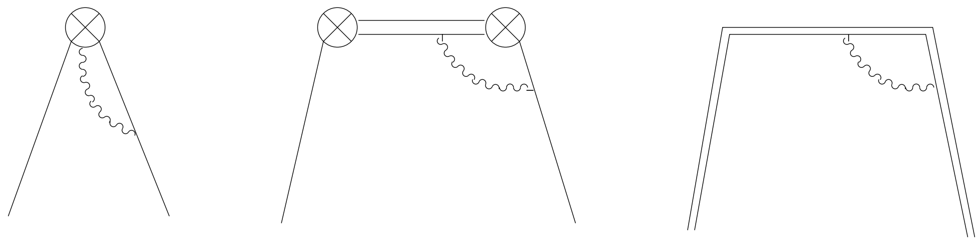

Figure 1: Different ways to compute, at 1-loop, the anomalous dimension of the twist two operators in field theory for large spin. The straight lines indicate scalar propagators, wavy lines denote gluons and double lines a Wilson line in the adjoint representation. $1c$ is just a Wilson line which can also be computed using the AdS/CFT.

supergravity picture, as macroscopic rotating strings whose energy turns out to be, for large S,

$$E \simeq S + \frac{\sqrt{g_s N}}{\pi} \ln S\,, \qquad (S \to \infty)\,. \tag{1.2}$$

In the CFT, E is interpreted as the conformal dimension Δ_S of the corresponding operator, and in fact, in the field theory, a similar behavior is obtained at one loop for the operators (1.1) except that the coefficient in front of $\ln S$ is linear in $g_s N$, where $g_s = g_{\rm YM}^2$ [6]. The field theory result is obtained by evaluating the expectation value

$$\Gamma^{[2]}_{\mathcal{O}_s} = \langle p|\mathcal{O}_s|p\rangle = C_S (ip_\mu \Delta^\mu)^S \left(\frac{\Lambda}{\mu}\right)^{\gamma_S}\,, \tag{1.3}$$

where $|p\rangle$ is a one particle Φ^I state with momentum p, Λ and μ are UV and IR cut offs and γ_S is the anomalous dimension ($\Delta_S = S + 2 + \gamma_S$). The operator $\mathcal{O}_s$ is defined as

$$\mathcal{O}_S(\Delta) = \mathrm{Tr}\left(\Phi^I \nabla_{\mu_1} \cdots \nabla_{\mu_S} \Phi^I\right) \Delta^{\mu_1} \cdots \Delta^{\mu_S}\,, \tag{1.4}$$

where Δ^μ is an arbitrary vector which is taken to be light-like so that only the traceless part contributes to $\mathcal{O}_s$. The Green function in eq. (1.3) can be evaluated perturbatively as in [9] and the result is that, at one loop, the leading contribution (for $S \to \infty$) to the anomalous dimension comes from the diagram in figure $1a$. which gives the logarithmic dependence $\Delta_S = S + 2 + \gamma_S$, $\gamma_S \sim \ln S$, $(S \to \infty)$. This behavior persist at two loops in supersymmetric and not supersymmetric theories.[3]

In [11], it was observed that the anomalous dimension, for large S, can also be computed by using Wilson loops. More precisely, that it is related to the anomalous dimensions of Wilson loops with cusps. This suggests an alternative method to use the AdS/CFT correspondence to perform the calculation which is to compute the relevant Wilson loop by using the results of [12]. In this paper we pursue this idea an obtain exactly the same result as with the rotating string, giving support to the identification between the operators $\mathcal{O}_S$ and the rotating strings.

[3]See [10] for a review of these results and references to the extensive literature in the subject.

In the next section 2 we review the calculation of [6] using rotating strings an explain how it can be done in an alternative way by using Wilson loops. In section 3 we perform the calculation of the Wilson loop by doing an analytic continuation of previous results in euclidean AdS space [13]. In section 4 we analyze these results and give its interpretation in terms of an euclidean word-sheet in $AdS_5 \times S_5$. In particular we show that the relevant world-sheet is completely determined by symmetry considerations. Finally, we give our conclusions in section 5.

2. Rotating strings and Wilson loops

In this section, after reviewing the calculation of the string rotating in AdS_5 [6] we summarize the argument of Korchemsky and Marchesini [11] which suggests an alternative computation using Wilson loops.

In [6] it was proposed that twist two operators corresponded to (semiclassical) rotating strings in AdS_5. The AdS_5 metric in global coordinates (see appendix) is

$$ds^2 = -g_{tt}dt^2 + d\rho^2 + g_{\phi\phi}d\Omega_3^2 \,, \tag{2.1}$$

with $g_{tt} = \cosh^2\rho$ and $g_{\phi\phi} = \sinh^2\rho$. A rotating string is given by $\rho = \sigma$, $t = \tau$ and $\phi = \omega\tau$ where ϕ is an angle around a maximal circle in the S^3. Using the Nambu-Goto action it is easy to compute the energy and angular momentum as functions of the parameter ω. In the limit $S \to \infty$ we have $\omega \to 1$ and

$$P_+ = E + \omega S = 4\frac{R^2}{2\pi\alpha'}\int_0^{\rho_0} d\rho \frac{g_{tt} + \omega^2 g_{\phi\phi}}{\sqrt{g_{tt} - \omega^2 g_{\phi\phi}}} \sim \int^{\rho_0} d\rho e^{2\rho} \sim e^{2\rho_0}\,, \tag{2.2}$$

$$P_- = E - \omega S = 4\frac{R^2}{2\pi\alpha'}\int_0^{\rho_0} d\rho\sqrt{g_{tt} - \omega^2 g_{\phi\phi}} \simeq 4\frac{R^2}{2\pi\alpha'}\int_0^{\rho_0} d\rho = \frac{R^2}{\pi\alpha'}2\rho_0\,, \tag{2.3}$$

where ρ_0 is such that $\omega = g_{tt}(\rho_0)/g_{\phi\phi}(\rho_0)$ and consequently $\rho_0 \to \infty$ as $\omega \to 1$. The approximations in (2.2), (2.3) are valid in that limit and, the one in (2.2) only up to a constant factor. From (2.2), (2.3), and using that $R^2/\alpha' = \sqrt{g_s N}$, we can derive (1.2) which, within this approximation can be written as

$$P_- \simeq \frac{\sqrt{g_s N}}{\pi}\ln P_+\,. \tag{2.4}$$

The approximate results can be thought of as arising from a string in a metric

$$ds^2 = e^{2\rho}dx_+dx_- + dx_-^2 + d\rho^2 + dy_{[2]}^2\,, \qquad 0 < \rho < \rho_0\,, \tag{2.5}$$

and extending along ρ and x_+. This metric is invariant under translations of x_- and under a combined translation in ρ and rescaling in x_+. This fact results in all segments $d\rho$ of the string contributing equally to P_- (in (2.3)) but weighted by $e^{2\rho}$ to P_+ (in (2.2)), resulting in $P_- \sim \ln P_+$. According to the rules of the AdS/CFT correspondence, the integrals in ρ should be interpreted in the field theory as sums of contributions from different scales and it would be interesting to see if their behavior can be derived by field theory considerations alone.

An alternative approach to computing the anomalous dimension is suggested by the fact that the operators $\mathcal{O}_S(\Delta)$ in eq. (1.4) arise in a powers series expansion of

$$W_{\Delta^\mu} = \mathrm{Tr}\left(\Phi^I(\Delta^\mu) e^{\int_0^\Delta A_\mu(t\Delta^\mu)\Delta^\mu dt} \Phi^I(0)\right) = \sum_{S=0}^{\infty} \frac{1}{S!} \mathcal{O}_S(\Delta)\,, \tag{2.6}$$

where the Wilson line is in the adjoint representation ($A_\mu = A_\mu^a C_{abc}$). Again we can compute

$$\langle p|W(\Delta^\mu)|p\rangle = \sum \frac{1}{S!} C_S (ip_\mu \Delta^\mu)^S \left(\frac{\Lambda}{\mu}\right)^{\gamma_S} . \tag{2.7}$$

Thus, the coefficient of $(p\Delta)^S$ in a power series expansion of $\langle p|W(\Delta^\mu)|p\rangle$ in $(p\Delta)$ determines the anomalous dimension of the operator $\mathcal{O}_S$. For example, a 1-loop calculation as in fig.1b gives the same result for γ_S as the one obtained from figure 1(a). In particular, the large S behavior of γ_S is related to the large $p\Delta$ behavior of $\langle p|W(\Delta^\mu)|p\rangle$.

It was argued in [11] that, in this limit, the relevant contribution to the anomalous dimension comes from soft gluons.[4] This fact allows the use of the eikonal approximation for the propagator of the external particles, in this case Φ. In Feynman diagrams, this amounts to replace the propagator of Φ with momentum $p-k$ by:

$$\frac{1}{(p-k)^2 + i\epsilon} \to \frac{1}{-2p.k + i\epsilon}\,, \tag{2.8}$$

where p is the momentum of the external Φ line ($p^2 = 0$) and k the momentum of an attached gluon line.

The calculation is then reduced to the computation of a Wilson line (W_M) as the one in fig.1c, where the Φ propagators are replaced by Wilson lines (in the adjoint representation) going from infinity to 0 and from $x^\mu = \Delta^\mu$ to infinity with momentum p and $-p$, respectively. These are joined by a Wilson line going from 0 to $x^\mu = \Delta^\mu$ which is the same as the one appearing in figure 1(b). The precise relation is that

$$\langle p|W(\Delta^\mu)|p\rangle \simeq e^{ip.\Delta} W_M\,, \tag{2.9}$$

where W_M is the Wilson line previously discussed.

The Wilson line has an anomalous dimension due to the fact that it possesses cusps [15]. As explained in [11] there is a subtlety because the expectation value of the Wilson loop diverges when part of it is light-like. If we regulate the Wilson loop by taking Δ^2 and p^2 non-vanishing, we expect, from [15], each cusp to contribute with

$$W_{\mathrm{cusp}} \sim \left(\frac{L}{\epsilon}\right)^{-\Gamma_{\mathrm{cusp}}(\gamma)}\,, \tag{2.10}$$

where L and ϵ are IR and UV cut offs, and the anomalous dimension $\Gamma_{\mathrm{cusp}}(\gamma)$ is only a function of the angle between p and Δ defined as

$$\cosh\gamma = \frac{p.\Delta}{\sqrt{p^2\Delta^2}}\,. \tag{2.11}$$

[4] See [14] for a detailed description of these ideas. We thank G. Korchemsky for clarifying this point to us.

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Since we are interested in the region where $p.\Delta$ is very large we should compute $\Gamma_{\rm cusp}(\gamma)$ for large values of γ. It was shown in [16] that, in that limit, the anomalous dimension behaves as

$$\Gamma_{\rm cusp}(\gamma) \simeq \bar{\Gamma}_{\rm cusp}|\gamma| \simeq \bar{\Gamma}_{\rm cusp}\ln(p.\Delta)\,, \qquad (\gamma\to\infty)\,, \tag{2.12}$$

where $\bar{\Gamma}_{\rm cusp}$ is a constant that depends only on $g_{\rm YM}$ and N. In the last approximation we kept only the leading contribution for large $(p.\Delta)$ which is independent of Δ^2 and p^2. Considering the contribution of both cusps we obtain that

$$\langle p|W(\Delta^\mu)|p\rangle \sim e^{ip.\Delta}\left(\frac{L}{\epsilon}\right)^{-2\bar{\Gamma}_{\rm cusp}\ln(p.\Delta)} = e^{ip.\Delta}(p.\Delta)^{-2\bar{\Gamma}_{\rm cusp}\ln(L/\epsilon)}, \tag{2.13}$$

where the approximation is valid for large $(p.\Delta)$. The fact that

$$\sum_{n=0}^{\infty} a_n\frac{x^n}{n!} \simeq x^\alpha e^x\,, \qquad (x\to\infty) \qquad \text{if} \quad a_n\sim n^\alpha\,, \qquad (n\to\infty), \tag{2.14}$$

allows us to conclude that

$$\langle p|W(\Delta^\mu)|p\rangle \sim \sum_S \frac{1}{S!}S^{-2\bar{\Gamma}_{\rm cusp}\ln(L/\epsilon)}(ip\Delta)^S = \sum_S\frac{1}{S!}\left(\frac{L}{\epsilon}\right)^{-2\bar{\Gamma}_{\rm cusp}\ln S}(ip\Delta)^S\,. \tag{2.15}$$

The result (2.14) can be derived by showing that, if we define a function $f(x)$ through

$$a_n = \int_0^1 s^n f(s)ds\,, \tag{2.16}$$

then both, the behavior of a_n for $n\to\infty$ and the behavior of

$$e^{-x}\sum a_n\frac{x^n}{n!} = \int_0^1 e^{(s-1)x}f(s)ds\,, \tag{2.17}$$

for $x\to\infty$, are determined by the properties of the function $f(s)$ near $s=1$.

Finally, comparing eq. (2.15) with eq. (2.7) we see that, after identifying $\Lambda/\mu\sim L/\epsilon$, the anomalous dimension of the operator $\mathcal{O}_S$ is equal to

$$\Delta_S - S - 2 = \gamma_S \simeq -2\bar{\Gamma}_{\rm cusp}\ln S\,, \qquad (S\to\infty)\,. \tag{2.18}$$

The above argument ignores the problem of setting $\Delta^2=p^2=0$ since in that case the terms discarded in eq. (2.12) are singular. We refer the reader to the original reference [11] for a more rigorous treatment which gives the same result (2.18).[5]

In the next section we obtain $\bar{\Gamma}_{\rm cusp}$ using the AdS/CFT correspondence to compute a Wilson loop with a cusp in Minkowski signature. This is simply an analytic continuation of the results of [13] where the calculation was done in Euclidean space. However, before that, we conclude this section by reviewing some properties of the Wilson loop cusp anomaly at one loop which will be useful to compare with the AdS/CFT result.

[5] In [11] the result was equivalently expressed in terms of the splitting function $P(z)$.

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For the $\mathcal{N} = 4$ case the one loop anomaly can be taken from [13], but its behavior with the angle is similar to the one in QCD [15]. If the cusp has an angle Θ (where $\Theta = \pi$ defines a straight line) the Wilson loop behaves as

$$\begin{aligned} W_E &= \left(\frac{L}{\epsilon}\right)^{-\Gamma_{\text{cusp}}(\Theta)}, \\ \Gamma_{\text{cusp}}(\Theta) &= \frac{g_s N}{4\pi^2}\left((\pi - \Theta)\cot\Theta + 1\right). \end{aligned} \tag{2.19}$$

The result (2.19) can be continued to Minkowski space by replacing $\Theta \to \pi + i\gamma$. The resulting $\Gamma^{(M)}_{\text{cusp}}(\gamma) = \Gamma_{\text{cusp}}(\pi + i\gamma)$ satisfies:

$$\begin{aligned} \Gamma^{(M)}_{\text{cusp}}(\gamma) &\simeq -\frac{g_s N}{6\pi^2}\gamma^2, \qquad (\gamma \to 0), \\ \Gamma^{(M)}_{\text{cusp}}(\gamma) &\simeq -\frac{g_s N}{4\pi^2}\gamma, \qquad (\gamma \to \infty), \end{aligned}$$

which then gives at 1-loop: $\bar{\Gamma}_{\text{cusp}} = -(g_s N)/4\pi^2$.

As an aside note that the euclidean result has the property

$$\Gamma_{\text{cusp}}(\Theta) \simeq \frac{g_s N}{4\pi^2}\frac{1}{\Theta}, \qquad (\Theta \to 0), \tag{2.20}$$

which can be understood from conformal invariance. In fact the flat metric

$$ds^2 = dr^2 + r^2 d\theta^2, \tag{2.21}$$

is invariant (up to a conformal factor) under the transformation $r \to r^\lambda$, $\theta \to \lambda\theta$. This alters the periodicity of θ and so, is not actually a conformal transformation. However, in the limit $\Theta \to 0$ we need to consider only small values of θ and the periodicity can be ignored implying that

$$W \sim \left(\frac{L}{\epsilon}\right)^{-\Gamma_{\text{cusp}}(\Theta)}, \tag{2.22}$$

is invariant only if $\Gamma_{\text{cusp}}(\Theta) \sim 1/\Theta$, as obtained in (2.20).

3. Cusp anomaly from supergravity.

The cusp anomaly can be computed by means of the AdS/CFT correspondence by computing the expectation value of a Wilson loop with a cusp following [12]. This computation was done in [13] in the Euclidean case.

In this section we summarize their calculation and perform an analytic continuation to Minkowski signature which allows us to compute $\bar{\Gamma}_{\text{cusp}}$ and, by plugging the result into (2.18), to reproduce (1.2).

In the next section we reobtain these results by a direct calculation and obtain the shape of the world-sheet ending in the boundary of AdS. One extra consideration is that here we need a Wilson loop in the adjoint representation. This corresponds to a quark and anti-quark propagating along the loop. We will consider simply a double cover of the surface, i.e. we include an extra factor of 2 at the end. Another point is that the Maldacena correspondence allows us only to compute Wilson lines of the type

$$W = \exp\left(i\int A_\mu \dot{x}^\mu + \Phi^I \theta_I |\dot{x}|\right). \tag{3.1}$$

For the section parallel to Δ^μ there is no problem since $|\Delta| = 0$. For the lines representing the incoming particles, instead of the operator (1.1) we will be computing the anomalous dimension of an operator which is a linear combination of that one and the one obtained by replacing Φ by the gauge field. All previous considerations still apply since (1.1) was taken only as an example. It will be interesting to further analyze this issue and see if supergravity can distinguish between the different twist two operators, or if it predicts that all of them have the same anomalous dimension for large S and large $g_s N$.

After these preliminary considerations, lets go back to the computation of the Wilson loop, starting by the Euclidean case of [13]. Consider Euclidean AdS space in Poincare coordinates where the metric is

$$ds^2 = \frac{1}{z^2}\left(dz^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2\right). \tag{3.2}$$

Taking polar coordinates (ρ,θ) in the plane (x_1,x_2) we will compute the area of a world-sheet which ends in the boundary $z = 0$ on two half-lines given by $\theta = \pm\Theta/2$ and $x_1 > 0$. By scale invariance the equation of the surface can be written as

$$z = \frac{\rho}{f(\theta)}, \tag{3.3}$$

with $f(\pm\Theta/2) = \infty$. Also, by symmetry, we should have that $(df/d\theta)(\theta = 0) = 0$. Extremizing the Nambu-Goto action for the string with respect to the function $f(\theta)$ we obtain the desired result (more details can be found in [13] or in the next section):

$$\begin{aligned}\Theta &= \int_{-\infty}^{+\infty} \frac{f_0\sqrt{1+f_0^2}}{(u^2+f_0^2)\sqrt{(1+u^2+f_0^2)(1+z^2+2f_0^2)}}du, \\ A &= \frac{R^2}{2\pi\alpha'}\ln\frac{L}{\epsilon}\int_{-\infty}^{+\infty}\left\{\sqrt{\frac{1+u^2+f_0^2}{1+u^2+2f_0^2}} - 1\right\}.\end{aligned} \tag{3.4}$$

Here A, is the desired area of the world sheet, given as a function of the angle Θ implicitly through the parameter f_0. The term -1 inside the integral is an infinite subtraction constant which makes the area finite. Also, L and ϵ are an infrared and ultraviolet cut-offs such that $\epsilon < \rho < L$. According to [12], $\ln W_M$ is equal to the area A and so, we identify the cusp anomaly as

$$\Gamma_{\text{cusp}}(\Theta) = -\frac{R^2}{2\pi\alpha'}\int_{-\infty}^{+\infty}\left\{\sqrt{\frac{1+u^2+f_0^2}{1+u^2+2f_0^2}} - 1\right\}. \tag{3.5}$$

For example, as a check, we can see that the limit $f_0 \to \infty$ corresponds to $\Theta \to 0$ and the cusp anomaly behaves as

$$\Gamma_{\text{cusp}}(\Theta) \simeq -\frac{c^2}{\Theta}, \qquad (\Theta \to 0). \tag{3.6}$$

We see the expected behavior $1/\Theta$ of eqns. (2.20) and (2.22). The constant c is given by

$$c=\int_{-\infty}^{\infty}\frac{du}{(1+u^2)^{3/2}(2+u^2)^{1/2}}\,. \tag{3.7}$$

Now we want to do an analytic continuation in $\Gamma_{\rm cusp}(\Theta)$ to $\Theta=\pi+i\gamma$ with γ real. It appears, from eq. (3.4), that if we analytically continue f_0 to imaginary values, Θ becomes purely imaginary. However, a singularity develops in the integral at $u=f_0$ which gives the extra π. Indeed, to handle the singularities we can perform the analytic continuation more carefully as in figure 2. Thus, we obtain a recipe to avoid the poles at $u=\pm f_0$. The result is:

$$\Theta=i\int_{-\infty}^{+\infty}\frac{f_0\sqrt{1-f_0^2}}{(u^2-f_0^2+i\epsilon)\sqrt{(1+u^2-f_0^2)(1+u^2-2f_0^2)}}du\,. \tag{3.8}$$

We can separate the real and imaginary part by doing explicitly the integrals in the small half circles around the poles obtaining

$$\begin{aligned}\Theta&=\pi+i\gamma\,,\\ \gamma&=\text{P.P.}\int_{-\infty}^{+\infty}\frac{f_0\sqrt{1-f_0^2}}{(u^2-f_0^2)\sqrt{(1+u^2-f_0^2)(1+u^2-2f_0^2)}}du\,,\end{aligned} \tag{3.9}$$

where γ is real (as long as $|f_0|<1/\sqrt{2}$) and given by the principal part of the integral. Finally, from (3.5), the cusp anomaly is given by

$$\Gamma_{\rm cusp}(\gamma)=-\sqrt{g_sN}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\{\sqrt{\frac{1+u^2-f_0^2}{1+u^2-2f_0^2}}-1\right\}. \tag{3.10}$$

We can consider first the limit $\gamma\to0$ which corresponds to $f_0\to0$ (note that now $\Theta\to\pi$ so this limit is different from the limit $\Theta\to0$ considered before). The result is that

$$\begin{aligned}\gamma\simeq-\pi f_0\,,\qquad \Gamma_{\rm cusp}(\gamma)&\simeq\sqrt{g_sN}\frac{1}{4}f_0^2\,,\qquad (f_0\to0)\\ \Rightarrow\Gamma_{\rm cusp}(\gamma)&\simeq\sqrt{g_sN}\frac{\gamma^2}{4\pi^2}\,,\qquad\qquad (\gamma\to\infty)\,,\end{aligned} \tag{3.11}$$

which has a similar behavior as the one loop result albeit with a different coefficient.

For us it is more interesting the behavior of $\Gamma_{\rm cusp}(\gamma)$ when $\gamma\to\infty$. As we noticed before we need $f_0<1/\sqrt{2}$. We can see easily from the integral in eq. (3.9) that the limit $f_0\to1/\sqrt{2}$ corresponds precisely to $|\gamma|\to\infty$. In this limit we have

$$\begin{aligned}\gamma\qquad&\simeq\sqrt{2}\ln\delta\,,\qquad\qquad \Gamma_{\rm cusp}(\gamma)\simeq\frac{1}{2\pi}\frac{1}{\sqrt{2}}\ln\delta\,,\ \left(f_0\to\frac{1}{\sqrt{2}}\right),\\ \Rightarrow\Gamma_{\rm cusp}(\gamma)&\simeq-\sqrt{g_sN}\frac{1}{2\pi}\frac{1}{2}|\gamma|\ \Rightarrow\ \bar{\Gamma}_{\rm cusp}\quad=-\sqrt{g_sN}\frac{1}{4\pi}\ \ (\gamma\to-\infty)\,,\end{aligned} \tag{3.12}$$

where $1-2f_0^2=2\delta$ and the limiting behaviors are obtained by considering the contribution to the integrals from the region $u\to0$. For example for γ we have

$$\begin{aligned}\gamma&\simeq\int\frac{du}{(2u^2-1)\sqrt{(1/2)+u^2}\sqrt{u^2+2\delta}}\\ &\simeq-\sqrt{2}\int_{-\epsilon}^{\epsilon}\frac{du}{\sqrt{u^2+2\delta}}=-2\sqrt{2}\text{arcsinh}\left(\frac{\epsilon}{\sqrt{2\delta}}\right)\simeq\sqrt{2}\ln\delta\,,\end{aligned} \tag{3.13}$$

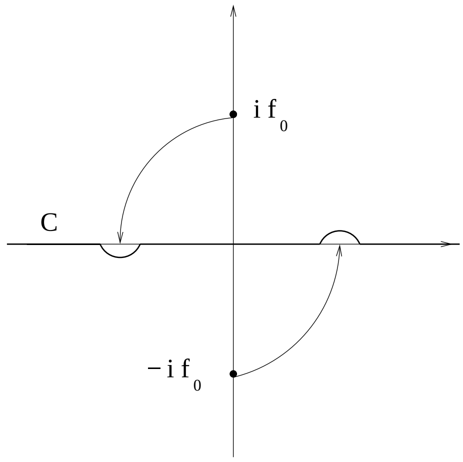

Figure 2: Rotation of f_0 in the complex plane necessary to get an imaginary angle Θ.

where $\delta \ll \epsilon \ll 1$ and we kept only the leading dependence in δ (for $\delta \to 0$). The calculation for $\Gamma_{\rm cusp}(\gamma)$ is similar.

Again, the behavior is similar to that of the one loop result but with a different coefficient. We can plug the result (3.12) into eq. (2.18) obtaining

$$\gamma_S = -2\bar{\Gamma}^{(\rm adj.)}_{\rm cusp} \ln S = \frac{\sqrt{g_s N}}{\pi} \ln S\,, \tag{3.14}$$

where, as explained at the beginning of this section, an extra factor of 2 is included, i.e. we took $\bar{\Gamma}^{(\rm adj.)}_{\rm cusp} = 2\bar{\Gamma}_{\rm cusp} = -\sqrt{g_s N}/2\pi$, to account for the fact that the Wilson loop should be in the adjoint representation. We see that the result agrees with that obtained from the rotating string calculation (1.2). First, the linear dependence with $\ln S$ agrees. This follows from the non-trivial fact that the Wilson loop in supergravity diverges linearly in γ for large γ. The dependence on the coupling constant also agrees but that was expected since the area of a Wilson loop scales as R^2. On the other hand also the numerical coefficient is the same giving support to the identification between twist two operators and rotating strings in AdS space. In the next section we will see, that in our case, the coefficient follows from the area of the limiting surface (at $f_0 = 1/\sqrt{2}$) which can be determined by symmetry considerations alone. This gives hope that the result can be understood directly in the field theory using symmetry arguments.

4. Wilson loop in Minkowski signature

In this section we use the AdS/CFT correspondence to compute the cusp anomaly directly in Minkowski signature.[6] This will give an interpretation to eqns. (3.9) and (3.10) in terms of an Euclidean world-sheet in AdS_5. We also analyze the surface corresponding to the limiting case $f_0 = 1/\sqrt{2}$ and show that it can be found using only symmetry arguments.

[6]See [17] for another example of a calculation of Wilson loops in Minkowski space.

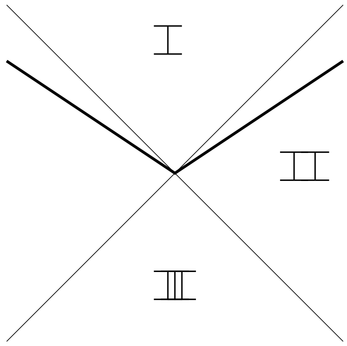

Figure 3: Wilson loop in the form of a wedge (thick lines). The thin lines show the light cone.

Consider a Wilson loop with a cusp as the one depicted in figure 3. In region II, we can use coordinates (ρ, ξ)[7] defined by

$$\begin{aligned} x &= \rho \cosh \xi \,, \\ t &= \rho \sinh \xi \,, \end{aligned} \tag{4.1}$$

which simplify the equations of motion. These coordinates can be extended to region I if we consider ρ to be purely imaginary and $\xi = i(\pi/2) + \tilde{\xi}$ with $\tilde{\xi}$ real. As before we can use an ansatz

$$z = \frac{\rho}{f(\xi)} \,. \tag{4.2}$$

Given the above properties of the coordinates (ρ,ξ) and the fact that z is real, we need $f(\xi)$ to be purely imaginary when ξ is in region I and also to vanish on the light cone ($\xi \to \infty$), since there ρ vanishes. These are complications introduced by the choice of coordinates, perhaps at this point it is useful to point out that the resulting surface has a simple shape as shown in figure 4. One may think that these complications could have been avoided by considering a Wilson line contained within region I. However in that case and for large angles, there is no surface in AdS space ending on such a Wilson line.

Since the world-sheet ends perpendicularly to the boundary, it has to be euclidean.

The action for such a world-sheet is given (in region II) by

$$\begin{aligned} S_E^{II} &= \int \frac{d\rho}{\rho} d\xi \sqrt{f'^2 - f^2 - f^4} \\ &= \ln \frac{L}{\epsilon} \int d\xi \sqrt{f'^2 - f^2 - f^4} \,. \end{aligned} \tag{4.3}$$

The fact that the integrand does not depend explicitly on ξ implies a conservation law:

$$\frac{f'^2}{\sqrt{f'^2 - f^2 - f^4}} - \sqrt{f'^2 - f^2 - f^4} = C \,, \tag{4.4}$$

[7]These are coordinates in the boundary of AdS space. ρ should not be confused with the radial direction in AdS space as used before.

with C a constant. The solution is given then by

$$\xi = \frac{1}{2}\gamma + f_0\sqrt{1-f_0^2}\int_f^\infty \frac{df}{f\sqrt{(1+f^2)(f^2+f_0^2)(1+f^2-f_0^2)}}\,, \tag{4.5}$$

where for later convenience we introduce f_0 through the relation

$$C^2 = f_0^2 - f_0^4\,, \qquad f_0^2 < 1\,. \tag{4.6}$$

Furthermore, for $f \to \infty$, namely $z \to 0$, we have $\xi \to \gamma/2$, i.e. the world-sheet ends on the line $(x,t) = \lambda(\cosh\gamma/2, \sinh\gamma/2)$. Normally, we would continue the surface until $f' = 0$ where we would match with the other half ending on $(x,t) = \lambda(-\cosh\gamma/2, \sinh\gamma/2)$. Here, f' can be 0 only in region I, so we need to continue the boundary coordinates into that region. First, we observe that, at $f = 0$, the integral diverges implying that $x/t = \tanh\xi \to 1$, namely, in the boundary, we approach the light-cone. On the other hand $\rho = \sqrt{x^2-t^2}$ also vanishes. We can compute the value of z by taking the limit $(\xi \to \infty)$ with x fixed:

$$z = \lim_{\xi\to\infty}\frac{\rho}{f(\xi)} = x\lim_{\xi\to\infty}\frac{\sqrt{1-\tanh^2\xi}}{2e^B e^{-\xi}} = x\,e^{-B} = t\,e^{-B}\,, \tag{4.7}$$

where B is a constant given by

$$B = \frac{\gamma}{2} + \int_0^\infty \frac{df}{f\sqrt{1+f^2}}\left\{\frac{f_0\sqrt{1-f_0^2}}{\sqrt{(f^2+f_0^2)(1+f^2-f_0^2)}} - 1\right\}. \tag{4.8}$$

To continue beyond the light cone (in the boundary) we have to take f imaginary, then, in region I, f should extend up to if_0 where $f' = 0$ to match the other half of the surface. Although this may seem strange we want to emphasize again that the surface is perfectly smooth, the analytic continuation is necessary only because of the coordinates we are using. This is most easily done by defining a variable $u^2 = f^2 + f_0^2$ which extends from 0 to ∞, meaning that f extends from if_0 to 0 along the imaginary axis and from 0 to ∞ along the real axis. In this way, and considering also the corresponding formula for the area, we reproduce the formulas (3.9) and (3.10).

It is interesting to study the limiting surface obtained when $f_0 \to 1/\sqrt{2}$. In fact in that case $f' = 0$ and the surface is simply

$$z = \sqrt{2}\rho = \sqrt{2}\sqrt{t^2-x^2}\,. \tag{4.9}$$

This surface ends on the light cone $t = \pm x$, $t > 0$ and extends only for $t > x$. However since $z > \rho$ it is always outside the AdS light cone $z = \rho$. In fact one can see that a surface $z = \alpha\rho$, $\alpha > 1$ has zero area for $\alpha = 1$ and $\alpha = \infty$ which corresponds to a light cone and two light-like planes respectively. That implies that the area will have an extreme for an intermediate value which is precisely $\alpha = \sqrt{2}$. Note also that the surface cannot be continued to Euclidean space. That will give $z^2 = -2(t^2+x^2)$ which has no solutions.[8]

[8]However, one can also continue $z \to iz$ and obtain a surface in de Sitter space.

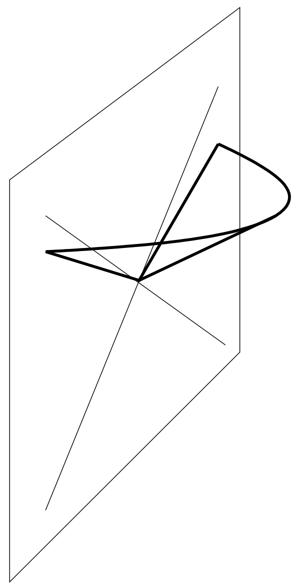

Figure 4: Sketch of the surface (in AdS_5) used to compute the Wilson loop of figure 3.

It is interesting to study this surface in global coordinates. If we take coordinates U, V, W, X, Y, Z such that

$$U^2 + V^2 - X^2 - Y^2 - Z^2 - W^2 = R^2 \,, \tag{4.10}$$

and use the relation between these and Poincare coordinates that we give in the appendix, we can see that the surface is given by

$$V^2 - X^2 = \frac{R^2}{2} \,, \qquad U^2 - W^2 = \frac{R^2}{2} \,, \qquad Y = Z = 0 \,. \tag{4.11}$$

This surface is in fact completely determined by the symmetries if one assumes that it is unique. Indeed, the light cone where the surface ends is invariant under scale transformations and boosts in (x, t). This corresponds in global coordinates to boosts in (U, W) and (V, X), respectively (see appendix) and implies that the equation determining the surface should be of the form

$$f(U^2 - W^2, V^2 - X^2) = 0 \,, \tag{4.12}$$

since reflection symmetry in Y and Z implies $Y = 0$, $Z = 0$. On the other hand, $U^2 - W^2 + V^2 - X^2 = R^2$ and so (4.12) can only be of the form

$$\begin{aligned} V^2 - X^2 &= a\frac{R^2}{2} \,, \\ U^2 - W^2 &= (1-a)\frac{R^2}{2} \,. \end{aligned} \tag{4.13}$$

Since there is an extra symmetry that interchanges (V, X) with (U, W), unless $a = 1/2$ there would be two different surfaces. Assuming that the surface is unique determines $a = 1/2$.

Returning to Poincare coordinates, we can see that the induced metric on the surface is

$$ds^2 = \frac{R^2}{2}\left(\frac{d\rho^2}{\rho^2} + d\xi^2\right), \tag{4.14}$$

and the area is given by

$$A = \frac{R^2}{2\pi\alpha'}\frac{1}{2}\int \frac{d\rho}{\rho}\int_{-\infty}^{+\infty} d\xi\,, \tag{4.15}$$

which diverges. However, performing the ρ integral between ϵ and L as before and the ξ integral between $-\gamma/2$ and $\gamma/2$ we get

$$A = \frac{R^2}{2\pi\alpha'}\frac{1}{2}\gamma \ln\frac{L}{\epsilon}\,, \tag{4.16}$$

which implies that $\bar{\Gamma}_{\rm cusp} = -\sqrt{g_s N}/4\pi$ as in (3.12). We see that the value $a = 1/2$ is the one that ultimately determines the anomalous dimension and, as we discussed, is fixed by the requirement that the solution be unique.

5. Conclusions

We have computed the anomalous dimension of twist two operators, in $\mathcal{N}=4$ SYM, in the limit of large angular momentum and large 't Hooft coupling by using Wilson loops in Minkowski space. The results are in agreement with the ones of [6] which where obtained by computing the energy-angular momentum relation of a semiclassical string rotating in AdS space. The agreement is not surprising since both calculations are done using the AdS/CFT correspondence but tests the identification between those operators and the rotating strings.

Furthermore the euclidean world-sheet that determines the value of the Wilson loop and the anomalous dimension is uniquely determined by the symmetries of the problem. Those are isometries of AdS space and so, correspond to conformal transformations in the SYM theory. It would be interesting if this can be done directly in the field theory, particularly in view of the fact that certain Wilson loops can be computed in $\mathcal{N}=4$ SYM and agree with the AdS/CFT result [18]. In our case a short calculation suggests that, near a cusp, summing ladder diagrams as in [18] is not enough to obtain the anomalous dimension.

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A. Coordinates in AdS_5

In the text various standard coordinates were used to parameterize AdS_5. For completeness we give here the corresponding definitions. Introducing cartesian coordinates (X,Y,Z,W,U,V) in $R^{4,2}$, AdS_5 is the manifold defined by the constraint:

$$-X^2 - Y^2 - Z^2 - W^2 + U^2 + V^2 = R^2\,, \tag{A.1}$$

for some arbitrary constant R. The metric is the one induced by the one in $R^{4,2}$:

$$ds^2 = dX^2 + dY^2 + dZ^2 + dW^2 - dU^2 - dV^2\,. \tag{A.2}$$

Global coordinates $(\rho, t, \theta_{1,2,3})$ as used in eq. (2.1) are defined by

$$\begin{aligned} &X = R\sinh\rho\cos\theta_1\,, && Y = R\sinh\rho\sin\theta_1\cos\theta_2\,, && Z = R\sinh\rho\sin\theta_1\sin\theta_2\cos\theta_3\,,\\ &W = R\sinh\rho\sin\theta_1\sin\theta_2\sin\theta_3\,, && U = R\cosh\rho\cos t\,, && V = R\cosh\rho\sin t\,. \end{aligned} \tag{A.3}$$

The metric is given by

$$ds^2 = R^2\left(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_{[3]}^2\right), \tag{A.4}$$

where $\Omega_{[3]}$ is a 3-sphere parameterized by $\theta_{1,2,3}$. The radial coordinate ρ extends from 0 to ∞. It is also useful to introduce another radial coordinate $0 < \xi < \pi/2$ through the relation $\cosh\rho = 1/\cos\xi$. The metric changes to

$$ds^2 = \frac{R^2}{\cos^2\xi}\left(-dt^2 + d\xi^2 + \sin^2\xi d\Omega_{[3]}^2\right). \tag{A.5}$$

In these coordinates the surface $V^2 - X^2 = R^2/2$ discussed in the text is described by the equation

$$\sin^2 t = \frac{1}{2}\cos^2\xi + \sin^2\xi\cos^2\theta_1\,. \tag{A.6}$$

Poincare coordinates $(t, x_{1,2,3}, z)$ cover only half of AdS_5 and are defined by

$$\begin{aligned} X &= R\frac{x_1}{z}\,, & Y &= R\frac{x_2}{z}\,, & Z &= R\frac{x_3}{z}\,,\\ W &= -\frac{1}{2z}(-R^2 + z^2 + x_i^2 - t^2)\,, & U &= \frac{1}{2z}(R^2 + z^2 + x_i^2 - t^2)\,, & V &= R\frac{t}{z}\,, \end{aligned} \tag{A.7}$$

with a metric

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + dx_i^2 - dt^2\right). \tag{A.8}$$

In the text we used the fact that a boost in direction x_1 corresponds to a boost in directions X, V and a scale transformation to a boost in directions W, U. The first property is obvious since U and W are invariant under a boost along x_1 and the only change is in X, V, being precisely the same boost as in x_1, t. To see the second property we have simply to notice that

$$U_+ = U + W = \frac{R^2}{z}\,, \qquad U_- = U - W = \frac{1}{z}(z^2 + x_i^2 - t^2)\,, \tag{A.9}$$

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transform as $U_+ \to \lambda U_+$, $U_- \to U_-/\lambda$, i.e. a boost, if we rescale all coordinates by a factor λ. Finally the equation of the surface discussed in the text:

$$z^2 = 2(x^2 - t^2)\,, \tag{A.10}$$

is equivalent to

$$\frac{R^4}{U_+^2} = 2\left(\frac{V^2R^2}{U_+^2} - \frac{X^2R^2}{U_+^2}\right) \qquad \Rightarrow \qquad V^2 - X^2 = \frac{R^2}{2}\,, \tag{A.11}$$

as used in section 4.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[4] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[5] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and PP waves from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021].

[6] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys.* **B 636** (2002) 99 [hep-th/0204051].

[7] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$*, *J. High Energy Phys.* **06** (2002) 007 [hep-th/0204226].

[8] J.G. Russo, *Anomalous dimensions in gauge theories from rotating strings in $AdS_5 \times S^5$*, *J. High Energy Phys.* **06** (2002) 038 [hep-th/0205244].

[9] H. Georgi and H.D. Politzer, *Electroproduction scaling in an asymptotically free theory of strong interactions*, *Phys. Rev.* **D 9** (1974) 416;
D.J. Gross and F. Wilczek, *Asymptotically free gauge theories, 2*, *Phys. Rev.* **D 9** (1974) 980.

[10] M. Axenides, E. Floratos and A. Kehagias, *Scaling violations in Yang-Mills theories and strings in AdS_5*, hep-th/0210091.

[11] G.P. Korchemsky, *Asymptotics of the Altarelli-Parisi-Lipatov evolution kernels of parton distributions*, *Mod. Phys. Lett.* **A 4** (1989) 1257;
G.P. Korchemsky and G. Marchesini, *Structure function for large x and renormalization of Wilson loop*, *Nucl. Phys.* **B 406** (1993) 225 [hep-ph/9210281].

[12] J.M. Maldacena, *Wilson loops in large-N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [hep-th/9803002];
S.-J. Rey and J. Yee, *Macroscopic strings as heavy quarks in large-N gauge theory and anti-de Sitter supergravity*, *Eur. Phys. J.* **C 22** (2001) 379 [hep-th/9803001].

JHEP12(2002)024

[13] N. Drukker, D.J. Gross and H. Ooguri, *Wilson loops and minimal surfaces*, *Phys. Rev.* **D 60** (1999) 125006 [hep-th/9904191].

[14] J.C. Collins, *Sudakov form factors*, in *Perturbative quantum chromodynamics*, A.H. Mueller ed., Advanced Series on Directions in High Energy Physics, **5**, World Scientific 1989.

[15] A.M. Polyakov, *Gauge fields as rings of glue*, *Nucl. Phys.* **B 164** (1980) 171.

[16] G.P. Korchemsky and A.V. Radyushkin, *Renormalization of the wilson loops beyond the leading order*, *Nucl. Phys.* **B 283** (1987) 342.

[17] A.A. Tseytlin and K. Zarembo, *Wilson loops in $N = 4$ SYM theory: rotation in S^5*, hep-th/0207241.

[18] J.K. Erickson, G.W. Semenoff and K. Zarembo, *Wilson loops in $N = 4$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 582** (2000) 155 [hep-th/0003055].

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